Proceedings of the Jangjeon Mathematical Society
28 (2025), No. 4, pp. 557 - 567

www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2025.28.4.012

Structural Insights into Mock Theta-Function Identities via Multivariable $\mathscr{R}$-Functions and Partition Theory

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Abstract

This paper introduces five new identities involving eighth-order mock theta functions, expressed in terms of multivariable $\mathscr{R}$-functions. These identities are derived using the well-known q_ι-product identities in conjunction with the classical Jacobi triple product identity. The results establish profound links between theta function representations and combinatorial partition theory. Specifically, we present key findings that offer structural insight into mock theta function identities, using multivariate $\mathscr{R}$-functions and partition theory. These discoveries uncover new patterns and identities within the framework of mock theta functions, with a particular focus on their multivariable structure and their connections to partition theory. In addition, we concisely summarize recent advancements in the field and discuss potential applications. The findings emphasize a significant relationship between our results and partition-theoretic identities.

2000 Mathematics Subject Classification. Primary 30C45; 30C50; 30C80; Secondary 11B65; 47B38.

Key Words and Phrases. Theta functions; multivariable $\mathscr{R}$-functions; Jacobi triple product identity; Ramanujan-type identities; q_ι-series; Rogers–Ramanujan continued fractions; partition theory; combinatorics.

1. INTRODUCTION AND DEFINITIONS

In this paper, we adopt the standard notation: $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{C}$ represent the sets of natural numbers, integers, and complex numbers, respectively. We also define

$$\mathbb{N}_0 := \mathbb{N} \cup \{0\} = \{0, 1, 2, \dots\}.$$

To express the content clearly and uniquely, q_ι has been used in place of q.

Date: Submitted on October 15, 2025.

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We adopt the conventional q_ι-series notation, consistent with the conventions outlined in [17, 18].
The q_ι-*shifted factorial* $(a;q_\iota)_n$ is defined for $|q_\iota|<1$ in the following manner:

$$(a;q_\iota)_n := \begin{cases} 1 & \text{if } n=0, \\ \prod_{k=0}^{n-1}(1-aq_\iota^k) & \text{if } n\in\mathbb{N}. \end{cases} \tag{1.1}$$

Let $a, q_\iota \in \mathbb{C}$, and assume, unless explicitly mentioned, that a is not equal to q_ι^{-m} for any $m\in\mathbb{N}_0$. The infinite q_ι-shifted factorial is defined as follows:

$$(a;q_\iota)_\infty := \prod_{k=0}^{\infty}(1-aq_\iota^k) = \prod_{k=1}^{\infty}(1-aq_\iota^{k-1}), \quad (a, q_\iota \in \mathbb{C};\ |q_\iota|<1). \tag{1.2}$$

Observe that the infinite product in equation (1.2) does not converge when $a\neq 0$ and $|q_\iota|\geq 1$. Thus, throughout this paper, whenever the expression $(a;q_\iota)_\infty$ appears, it is implicitly assumed that $|q_\iota|<1$.

We adopt the following notational conventions throughout:

$$(a_1, a_2, \ldots, a_m; q_\iota)_n := (a_1;q_\iota)_n(a_2;q_\iota)_n\cdots(a_m;q_\iota)_n, \tag{1.3}$$

$$(a_1, a_2, \ldots, a_m; q_\iota)_\infty := (a_1;q_\iota)_\infty(a_2;q_\iota)_\infty\cdots(a_m;q_\iota)_\infty. \tag{1.4}$$

The general theta function $f(a,b)$ was introduced by Ramanujan [15, 14] and is defined in the following manner [5, 20]:

$$\begin{aligned} f(a,b) &= 1+\sum_{n=1}^{\infty}(ab)^{\frac{n(n-1)}{2}}(a^n+b^n) \\ &= \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} = f(b,a), \quad (|ab|<1). \end{aligned} \tag{1.5}$$

It can be deduced from equation (1.5) that

$$f(a,b) = a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} f(a(ab)^n, b(ab)^{-n}) = f(b,a), \quad (n\in\mathbb{Z}). \tag{1.6}$$

In addition, Ramanujan rediscovered Jacobi's celebrated triple product identity, which he expressed in the following form [5, 7, 20]:

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{1.7}$$

Alternatively, using the notation in [12], it may be expressed as:

$$\begin{aligned} \sum_{n=-\infty}^{\infty} q_\iota^{n^2} z^n &= \prod_{n=1}^{\infty}(1-q_\iota^{2n})(1+zq_\iota^{2n-1})\left(1+\frac{q_\iota^{2n-1}}{z}\right) \\ &= (q_\iota^2;q_\iota^2)_\infty(-zq_\iota;q_\iota^2)_\infty\left(-\frac{q_\iota}{z};q_\iota^2\right)_\infty, \quad (|q_\iota|<1; z\neq 0). \end{aligned} \tag{1.8}$$

Remark 1.1. Equation (1.6) is valid only when $n\in\mathbb{Z}$. If n is not an integer, the formula gives only an approximate value. For a detailed explanation, see [15].

Remark 1.2. The identity of q_ι-series given in equation (1.7) (or its equivalent form above) was originally discovered by Carl Friedrich Gauss in the early 19^{th} century.

Many significant identities from the q_ι-series are derived naturally from the Jacobi triple product identity given in (1.7). Notable examples include [5]:

$$\begin{aligned}\varphi(q_\iota) &:= \sum_{n=-\infty}^{\infty} q_\iota^{n^2} = 1 + 2\sum_{n=1}^{\infty} q_\iota^{n^2} \\ &= \left\{(-q_\iota; q_\iota^2)_\infty\right\}^2 (q_\iota^2; q_\iota^2)_\infty = \frac{(-q_\iota; q_\iota^2)_\infty (q_\iota^2; q_\iota^2)_\infty}{(q_\iota; q_\iota^2)_\infty (-q_\iota^2; q_\iota^2)_\infty}.\end{aligned} \tag{1.9}$$

Another classical identity is given by

$$\psi(q_\iota) := f(q_\iota, q_\iota^3) = \sum_{n=0}^{\infty} q_\iota^{\frac{n(n+1)}{2}} = \frac{(q_\iota^2; q_\iota^2)_\infty}{(q_\iota; q_\iota^2)_\infty}. \tag{1.10}$$

Additionally, we have

$$\begin{aligned}f(-q_\iota) := f(-q_\iota, -q_\iota^2) &= \sum_{n=-\infty}^{\infty} (-1)^n q_\iota^{\frac{n(3n-1)}{2}} \\ &= \sum_{n=0}^{\infty} (-1)^n q_\iota^{\frac{n(3n-1)}{2}} + \sum_{n=1}^{\infty} (-1)^n q_\iota^{\frac{n(3n+1)}{2}} \\ &= (q_\iota; q_\iota)_\infty.\end{aligned} \tag{1.11}$$

Euler's Pentagonal Number Theorem is expressed in equation (1.11). Interestingly, the identity below provides its equivalent in analytic form:

$$(-q_\iota; q_\iota)_\infty = \frac{1}{(q_\iota; q_\iota^2)_\infty} = \frac{1}{\chi(-q_\iota)}, \tag{1.12}$$

where $\chi(q_\iota)$ denotes a modular character function. This formulation provides the analytic representation of Euler's classical result [2, 4].

Theorem 1 (Euler's Pentagonal Number Theorem). For every natural number n, the number of partitions of n into distinct integers, denoted by $\mathcal{D}(n)$, is precisely the same as the number of partitions of n into odd integers, represented by $\mathcal{O}(n)$; in other words, $\mathcal{D}(n) = \mathcal{O}(n)$.

We now introduce the continued Rogers–Ramanujan fraction $\mathscr{R}(q_\iota)$, defined as

$$\begin{aligned}\mathscr{R}(q_\iota) := q_\iota^{\frac{1}{5}} \frac{H(q_\iota)}{G(q_\iota)} = q_\iota^{\frac{1}{5}} \frac{f(-q_\iota, -q_\iota^4)}{f(-q_\iota^2, -q_\iota^3)} &= q_\iota^{\frac{1}{5}} \frac{(q_\iota; q_\iota^5)_\infty (q_\iota^4; q_\iota^5)_\infty}{(q_\iota^2; q_\iota^5)_\infty (q_\iota^3; q_\iota^5)_\infty} \\ &= \frac{q_\iota^{\frac{1}{5}}}{1+}\ \frac{q_\iota}{1+}\ \frac{q_\iota^2}{1+}\ \frac{q_\iota^3}{1+} \quad (|q_\iota| < 1).\end{aligned} \tag{1.13}$$

The functions $G(q_\iota)$ and $H(q_\iota)$, fundamental to the structure of identities similar to those of Rogers–Ramanujan, are introduced with the following definitions:

(1.14)
$$G(q_\iota) := \sum_{n=0}^{\infty} \frac{q_\iota^{n^2}}{(q_\iota; q_\iota)_n} = \frac{f(-q_\iota^5)}{f(-q_\iota, -q_\iota^4)} = \frac{1}{(q_\iota; q_\iota^5)_\infty (q_\iota^4; q_\iota^5)_\infty} = \frac{(q_\iota^2; q_\iota^5)_\infty (q_\iota^3; q_\iota^5)_\infty (q_\iota^5; q_\iota^5)_\infty}{(q_\iota; q_\iota)_\infty},$$

(1.15)
$$H(q_\iota) := \sum_{n=0}^{\infty} \frac{q_\iota^{n(n+1)}}{(q_\iota;q_\iota)_n} = \frac{f(-q_\iota^5)}{f(-q_\iota^2,-q_\iota^3)} = \frac{1}{(q_\iota^2;q_\iota^5)_\infty(q_\iota^3;q_\iota^5)_\infty} = \frac{(q_\iota;q_\iota^5)_\infty(q_\iota^4;q_\iota^5)_\infty(q_\iota^5;q_\iota^5)_\infty}{(q_\iota;q_\iota)_\infty}.$$

The functions $f(a,b)$ and $f(-q_\iota)$ that appear in preceding identities are defined in (1.5) and (1.11), respectively.

For further historical context and recent advancements related to the continued Rogers–Ramanujan fraction (1.13) and associated identities (1.14) and (1.15), along with related identities (1.14) and (1.15), one can consult the seminal reference [5]. Subsequent developments in [7, 19, 17]. Additional results on continued fractions are discussed in [6] and the references therein.

Theorem 2. For $|q_\iota| < 1$, the following continued fraction identity holds:

$$(q_\iota^2;q_\iota^2)_\infty(-q_\iota;q_\iota)_\infty = \frac{(q_\iota^2;q_\iota^2)_\infty}{(q_\iota;q_\iota^2)_\infty} = \frac{1}{1-}\,\frac{q_\iota}{1+}\,\frac{q_\iota(1-q_\iota)}{1-}\,\frac{q_\iota^3}{1+}\,\frac{q_\iota^2(1-q_\iota^2)}{1-}\,\frac{q_\iota^5}{1+}\,\frac{q_\iota^3(1-q_\iota^3)}{1-}\cdots$$

(1.16)
$$= \cfrac{1}{1-\cfrac{q_\iota}{1+\cfrac{q_\iota(1-q_\iota)}{1-\cfrac{q_\iota^3}{1+\cfrac{q_\iota^2(1-q_\iota^2)}{1-\cfrac{q_\iota^5}{1+\cfrac{q_\iota^3(1-q_\iota^3)}{1-\vdots}}}}}}},$$

$$\frac{(q_\iota;q_\iota^5)_\infty(q_\iota^4;q_\iota^5)_\infty}{(q_\iota^2;q_\iota^5)_\infty(q_\iota^3;q_\iota^5)_\infty} = \frac{1}{1+}\,\frac{q_\iota}{1+}\,\frac{q_\iota^2}{1+}\,\frac{q_\iota^3}{1+}\,\frac{q_\iota^4}{1+}\,\frac{q_\iota^5}{1+}\,\frac{q_\iota^6}{1+}\cdots$$

(1.17)
$$= \cfrac{1}{1+\cfrac{q_\iota}{1+\cfrac{q_\iota^2}{1+\cfrac{q_\iota^3}{1+\cfrac{q_\iota^4}{1+\cfrac{q_\iota^5}{1+\cfrac{q_\iota^6}{1+\vdots}}}}}}},$$

and

$$C(q_\iota) = \frac{(q_\iota^2;q_\iota^5)_\infty(q_\iota^3;q_\iota^5)_\infty}{(q_\iota;q_\iota^5)_\infty(q_\iota^4;q_\iota^5)_\infty} = 1 + \frac{q_\iota}{1+}\,\frac{q_\iota^2}{1+}\,\frac{q_\iota^3}{1+}\,\frac{q_\iota^4}{1+}\,\frac{q_\iota^5}{1+}\,\frac{q_\iota^6}{1+}\cdots$$

$$= 1 + \cfrac{q_\iota}{1 + \cfrac{q_\iota^2}{1 + \cfrac{q_\iota^3}{1 + \cfrac{q_\iota^4}{1 + \cfrac{q_\iota^5}{1 + \cfrac{q_\iota^6}{1 + \vdots}}}}}}. \tag{1.18}$$

In his 2015 work, Andrews [1] introduced novel double-sum hypergeometric q_ι-series representations for various partition families, highlighting the significance of double-series techniques in combinatorial partition identities. He proposed the generalized family $\mathscr{R}(k, l, s, t, p, q_\iota)$, presenting three special cases through multivariate $\mathscr{R}$-functions. Follow-up studies have extended these results, providing new generalizations in the domain of partition theory (see [7], [16], [21]). A particularly noteworthy contribution in this direction is Hahn's recent work [10].

This paper focuses on deriving two new identities that illuminate fundamental relationships among the functions $\mathscr{R}_\alpha$, $\mathscr{R}_\beta$, and $\mathscr{R}_m$, while also establishing their connections with q_ι-product identities.

$$\mathscr{R}(k, l, s, t, p, q_\iota) := \sum_{n=0}^{\infty} q_\iota^{k\binom{n}{2}+ln} \mathscr{R}(s, t, p, q_\iota; n), \tag{1.19}$$

where

$$r(s, t, p, q_\iota; n) = \sum_{j=0}^{\lfloor \frac{n}{t} \rfloor} (-1)^j \frac{q_\iota^{tp\binom{j}{2}+(q_\iota - st)_j}}{(q_\iota; q_\iota)_{n-tj}(q_\iota^{tp}; q_\iota^{tp})_j}. \tag{1.20}$$

Interested readers can refer to [20] for further details. We also recall some interesting special cases of the function defined in (1.19).

If $\mathscr{R}(k, l, s, t, p, q_\iota)$ is defined by the $\mathscr{R}$-function, then:

$$\mathscr{R}(2, 1, 1, 1, 2, 2) = (-q_\iota; q_\iota^2)_\infty. \tag{1.21}$$

If $\mathscr{R}(k, l, s, t, p, q_\iota)$ is defined by the $\mathscr{R}$-function as in equation (1.19), then:

$$\mathscr{R}(2, 2, 1, 1, 2, 2) = (-q_\iota^2; q_\iota^2)_\infty. \tag{1.22}$$

Furthermore, if $\mathscr{R}(k, l, s, t, p, q_\iota)$ is defined by the $\mathscr{R}$-function in equation (1.19), we obtain:

$$\mathscr{R}(m, m, 1, 1, 1, 2) = \frac{(q_\iota^{2m}; q_\iota^{2m})_\infty}{(q_\iota^m; q_\iota^{2m})_\infty}. \tag{1.23}$$

Recently, Srivastva [19] introduced the following notations:

$$\mathscr{R}_\alpha = \mathscr{R}(2, 1, 1, 1, 2, 2), \quad \mathscr{R}_\beta = \mathscr{R}(2, 2, 1, 1, 2, 2), \quad \mathscr{R}_m = \mathscr{R}(m, m, 1, 1, 1, 2), \quad m = 1, 2, 3, \ldots.$$

2. A Set of Lemmas

This section focuses on reviewing particular dissection formulas associated with eta-function quotients. The fundamental outcomes of this study rely on these formulas:

Lemma 2.1. *We have the following* 4*-dissections:*

$$\frac{U_2^3}{U_1^3} = \frac{U_6}{U_3} + 3q_\iota \frac{U_6^4 U_9^5}{U_3^8 U_{18}} + 6q_\iota^2 \frac{U_6^3 U_9^2 U_{18}^2}{U_3^7} + 12q_\iota^3 \frac{U_6^2 U_{18}^5}{U_3^6 U_9}. \tag{2.1}$$

The proof of equation (2.1) was provided by P. C. Toh [22].

Lemma 2.2. *We have the following* 2*-dissections:*

$$\frac{U_2^2}{U_1}=\frac{U_6U_9^2}{U_3U_{18}}+q_\iota\frac{U_{18}^2}{U_9}, \tag{2.2}$$

This dissection has been proven and discussed by several prominent mathematicians, including S. Cooper [8], G. E. Andrews and B. C. Berndt [3], and D. Hickerson [11].

Lemma 2.3. *We have the following* 2*-dissections:*

$$\frac{1}{U_1^2}=\frac{U_8^5}{U_2^5U_{16}^2}+2q_\iota\frac{U_4^2U_{16}^2}{U_2^5U_8}, \tag{2.3}$$

$$\frac{1}{U_1^4}=\frac{U_4^{14}}{U_2^{14}U_8^4}+4q_\iota\frac{U_4^2U_8^4}{U_2^{10}}. \tag{2.4}$$

The proofs of equations (2.3) and (2.4) were introduced by Xia and Yao [23], providing novel approaches.

Lemma 2.4. *We have the following* 3*-dissection:*

$$\frac{U_2}{U_1^2}=\frac{U_6^4U_9^6}{U_3^8U_{18}^3}+2q_\iota\frac{U_6^3U_9^3}{U_3^7}+4q_\iota^2\frac{U_6^2U_{18}^3}{U_3^6}. \tag{2.5}$$

Bruce C. Berndt discussed this dissection [5] in his work on Ramanujan's notebooks. His book examines many results from Ramanujan's work, including similar dissections related to theta functions.

We now use some pre-proven results to derive the desired outcomes:

$$16\frac{U_2^{15}U_8^2}{U_1^{13}U_4^3}+32\frac{U_2U_4^{11}}{U_1^9U_8^2}=48\frac{U_4^{39}}{U_1U_2^{27}U_8^{10}}+448q_\iota\frac{U_4^{27}}{U_1U_2^{23}U_8^2}+1280q_\iota^2\frac{U_4^{15}U_8^6}{U_1U_2^{19}}+1024q_\iota^3\frac{U_4^3U_8^{14}}{U_1U_2^{15}}. \tag{2.6}$$

$$4\frac{U_2^4U_4^5}{U_1^6U_8^2}=4\frac{U_4^{19}}{U_1^2U_2^{10}U_8^6}+16q_\iota\frac{U_4^7U_8^2}{U_1^2U_2^6}. \tag{2.7}$$

$$4\frac{U_2^4U_4^5}{U_1^6U_8^2}=4\frac{U_4^{19}}{U_2^{15}U_8U_{16}^2}+8q_\iota\frac{U_4^{21}U_{16}^2}{U_2^{15}U_8^7}+16q_\iota\frac{U_4^7U_8^7}{U_2^{11}U_{16}^2}+32q_\iota^2\frac{U_4^9U_8U_{16}^2}{U_2^{11}}. \tag{2.8}$$

$$8\frac{U_2^6U_8^2}{U_1^6U_4}=8\frac{U_4^{13}}{U_1^2U_2^8U_8^2}+32q_\iota\frac{U_4U_8^6}{U_1^2U_2^4}. \tag{2.9}$$

$$108\frac{U_2^{14}U_3^7}{U_1^{18}U_6^2}+324q_\iota\frac{U_2^6U_3^3U_6^6}{U_1^{14}}=108\frac{U_4^{14}U_3^7}{U_1^{14}U_6^2U_8^4}+324q_\iota\frac{U_4^{14}U_3^3U_6^6}{U_1^{10}U_2^8U_8^4}+432q_\iota\frac{U_4^2U_2^4U_3^7U_8^4}{U_1^{10}U_6^2}. \tag{2.10}$$

Proof of 2.6. Followed by [9], we can write it as:

$$\sum_{n=0}^{\infty}V(16n+14)q_\iota^n=16\frac{U_2^{15}U_8^2}{U_1U_4^3}\left(\frac{1}{U_1^4}\right)^3+32\frac{U_2U_4^{11}}{U_1U_8^2}\left(\frac{1}{U_1^4}\right)^2.$$

Putting the values of $(\frac{1}{U_1^4})$ from equation (2.4), we get the equation (2.6). □

Proof of 2.7. Followed by [9], for $n\geq 0$, we have

$$\sum_{n=0}^{\infty}V(8n+2)q_\iota^n=4\left(\frac{U_2^4U_4^5}{U_1^2U_8^2}\right)\left(\frac{1}{U_1^4}\right). \tag{2.11}$$

Putting the values of $(\frac{1}{U_1^4})$ from equation (2.4), we get the equation (2.7). □

Proof of 2.8. From [9], we can write it as:

$$\sum_{n=0}^{\infty} V(8n+2)q_\iota^n = 4\left(\frac{U_2^4U_4^5}{U_8^2}\right)\left(\frac{1}{U_1^2}\right)\left(\frac{1}{U_1^4}\right). \tag{2.12}$$

Putting the values of $(\frac{1}{U_1^2})$ and $(\frac{1}{U_1^4})$ from equation (2.3) and (2.4), we get

$$\sum_{n=0}^{\infty} V(8n+2)q_\iota^n = \left(\frac{U_2^4U_4^5}{U_8^2}\right)\left(\frac{U_8^5}{U_2^5U_{16}^2}+2q_\iota\frac{U_4^2U_{16}^2}{U_2^5U_8}\right)\left(\frac{U_4^{14}}{U_2^{14}U_8^4}+4q_\iota\frac{U_4^2U_8^4}{U_2^{10}}\right). \tag{2.13}$$

Simplify the right side of above equation to get the equation (2.8). □

Proof of 2.9. Followed by [9], we can write it as:

$$\sum_{n=0}^{\infty} V(8n+6)q_\iota^n = 8\left(\frac{U_2^6U_8^2}{U_1^2U_4}\right)\left(\frac{1}{U_1^4}\right). \tag{2.14}$$

Putting the values of $(\frac{1}{U_1^4})$ from equation (2.4) in the above equation, we get the equation (2.9). □

Proof of 2.10. Followed by [9], we can write it as:

$$\sum_{n=0}^{\infty} V(24n+19)q_\iota^n = 108\left(\frac{U_2^{14}U_3^7}{U_1^{14}U_6^2}\right)\left(\frac{1}{U_1^4}\right)+324q_\iota\left(\frac{U_2^6U_3^3U_6^6}{U_1^{10}}\right)\left(\frac{1}{U_1^4}\right). \tag{2.15}$$

Putting the values of $(\frac{1}{U_1^4})$ from equation (2.4), we get the equation (2.10). □

3. A Set of Main Results

Theorem 3.1.

$$\begin{aligned}
&16\frac{[\mathscr{R}_A]^7\mathscr{R}_D]}{(q_\iota;q_\iota)_\infty^6\mathscr{R}_B}+32\frac{[\mathscr{R}_A]^3[\mathscr{R}_B]^5}{(q_\iota;q_\iota)_\infty^6\mathscr{R}_D}=48\frac{[\mathscr{R}_B]^{17}}{(q_\iota;q_\iota)_\infty^6[\mathscr{R}_A]^5[\mathscr{R}_D]^5}+448q_\iota\frac{[\mathscr{R}_B]^{13}}{(q_\iota;q_\iota)_\infty^6[\mathscr{R}_A]^5\mathscr{R}_D}\\
&+1280q_\iota^2\frac{[\mathscr{R}_B]^9[\mathscr{R}_D]^3}{(q_\iota;q_\iota)_\infty^6[\mathscr{R}_A]^5}+1024q_\iota^3\frac{[\mathscr{R}_B]^5[\mathscr{R}_D]^7}{(q_\iota;q_\iota)_\infty^6[\mathscr{R}_A]^5}.
\end{aligned} \tag{3.1}$$

Proof. Converting the equation (2.6) in q_ι-pochhammer symbols:

$$\begin{aligned}
&16\frac{(q_\iota^2;q_\iota^2)_\infty^{15}(q_\iota^8;q_\iota^8)_\infty^2}{(q_\iota;q_\iota)_\infty^{13}(q_\iota^4;q_\iota^4)_\infty^3}+32\frac{(q_\iota^2;q_\iota^2)_\infty(q_\iota^4;q_\iota^4)_\infty^{11}}{(q_\iota;q_\iota)_\infty^9(q_\iota^8;q_\iota^8)_\infty^2}=48\frac{(q_\iota^4;q_\iota^4)_\infty^{39}}{(q_\iota;q_\iota)_\infty(q_\iota^2;q_\iota^2)_\infty^{27}(q_\iota^8;q_\iota^8)_\infty^{10}}+\\
&448q_\iota\frac{(q_\iota^4;q_\iota^4)_\infty^{27}}{(q_\iota;q_\iota)_\infty(q_\iota^2;q_\iota^2)_\infty^{23}(q_\iota^8;q_\iota^8)_\infty^2}+1280q_\iota^2\frac{(q_\iota^4;q_\iota^4)_\infty^{15}(q_\iota^8;q_\iota^8)_\infty^6}{(q_\iota;q_\iota)_\infty(q_\iota^2;q_\iota^2)_\infty^{19}}+1024q_\iota^3\frac{(q_\iota^4;q_\iota^4)_\infty^3(q_\iota^8;q_\iota^8)_\infty^{14}}{(q_\iota;q_\iota)_\infty(q_\iota^2;q_\iota^2)_\infty^{15}}.
\end{aligned} \tag{3.2}$$

Equation (3.2) can be written as:

$$\begin{aligned}
&16\frac{1}{(q_\iota;q_\iota)_\infty^6}\frac{(q_\iota^2;q_\iota^2)_\infty^{14}}{(q_\iota;q_\iota)_\infty^7}\frac{(q_\iota^8;q_\iota^8)_\infty^2}{(q_\iota^4;q_\iota^4)_\infty}\frac{(q_\iota^2;q_\iota^2)_\infty}{(q_\iota^4;q_\iota^4)_\infty^2}+32\frac{1}{(q_\iota;q_\iota)_\infty^6}\frac{(q_\iota^4;q_\iota^4)_\infty}{(q_\iota^8;q_\iota^8)_\infty^2}\frac{(q_\iota^2;q_\iota^2)_\infty(q_\iota^4;q_\iota^4)_\infty^{10}}{(q_\iota;q_\iota)_\infty^3}\\
&=48\frac{1}{(q_\iota;q_\iota)_\infty(q_\iota^2;q_\iota^2)_\infty^{10}}\frac{(q_\iota^4;q_\iota^4)_\infty^5}{(q_\iota^8;q_\iota^8)_\infty^{10}}\frac{(q_\iota^4;q_\iota^4)_\infty^{34}}{(q_\iota^2;q_\iota^2)_\infty^{17}}+448q_\iota\frac{1}{(q_\iota;q_\iota)_\infty(q_\iota^2;q_\iota^2)_\infty^{10}}\frac{(q_\iota^4;q_\iota^4)_\infty^{26}}{(q_\iota^2;q_\iota^2)_\infty^{13}}\frac{(q_\iota^4;q_\iota^4)_\infty}{(q_\iota^8;q_\iota^8)_\infty^2}\\
&1280q_\iota^2\frac{(q_\iota^4;q_\iota^4)_\infty^{14}}{(q_\iota^2;q_\iota^2)_\infty^7}\frac{(q_\iota^4;q_\iota^4)_\infty(q_\iota^8;q_\iota^8)_\infty^6}{(q_\iota;q_\iota)_\infty(q_\iota^2;q_\iota^2)_\infty^{12}}+1024q_\iota^3\frac{(q_\iota^4;q_\iota^4)_\infty^2}{(q_\iota^2;q_\iota^2)_\infty}\frac{(q_\iota^4;q_\iota^4)_\infty(q_\iota^8;q_\iota^8)_\infty^{14}}{(q_\iota;q_\iota)_\infty(q_\iota^2;q_\iota^2)_\infty^{14}}.
\end{aligned} \tag{3.3}$$

By using equation (1.23)(with $m = 1, 2, 3, 4, 8$)

$$\mathscr{R}_A = \mathscr{R}(1,1,1,1,1,2) = \frac{(q_\iota^2;q_\iota^2)_\infty}{(q_\iota;q_\iota^2)_\infty}.\frac{(q_\iota^2;q_\iota^2)_\infty}{(q_\iota^2;q_\iota^2)_\infty} = \frac{(q_\iota^2;q_\iota^2)_\infty^2}{(q_\iota;q_\iota)_\infty}, \tag{3.4}$$

$$\mathscr{R}_B = \mathscr{R}(2,2,1,1,1,2) = \frac{(q_\iota^4;q_\iota^4)_\infty}{(q_\iota^2;q_\iota^4)_\infty}.\frac{(q_\iota^4;q_\iota^4)_\infty}{(q_\iota^4;q_\iota^4)_\infty} = \frac{(q_\iota^4;q_\iota^4)_\infty^2}{(q_\iota^2;q_\iota^2)_\infty}, \tag{3.5}$$

$$\mathscr{R}_C = \mathscr{R}(3,3,1,1,1,2) = \frac{(q_\iota^6;q_\iota^6)_\infty}{(q_\iota^3;q_\iota^6)_\infty}.\frac{(q_\iota^6;q_\iota^6)_\infty}{(q_\iota^6;q_\iota^6)_\infty} = \frac{(q_\iota^6;q_\iota^6)_\infty^2}{(q_\iota^3;q_\iota^3)_\infty}, \tag{3.6}$$

$$\mathscr{R}_D = \mathscr{R}(4,4,1,1,1,2) = \frac{(q_\iota^8;q_\iota^8)_\infty}{(q_\iota^4;q_\iota^8)_\infty}.\frac{(q_\iota^8;q_\iota^8)_\infty}{(q_\iota^8;q_\iota^8)_\infty} = \frac{(q_\iota^8;q_\iota^8)_\infty^2}{(q_\iota^4;q_\iota^4)_\infty}, \tag{3.7}$$

$$\mathscr{R}_E = \mathscr{R}(8,8,1,1,1,2) = \frac{(q_\iota^{16};q_\iota^{16})_\infty}{(q_\iota^8;q_\iota^{16})_\infty}.\frac{(q_\iota^{16};q_\iota^{16})_\infty}{(q_\iota^{16};q_\iota^{16})_\infty} = \frac{(q_\iota^{16};q_\iota^{16})_\infty^2}{(q_\iota^8;q_\iota^8)_\infty}. \tag{3.8}$$

Thus, by applying (3.4), (3.5) and (3.7), we get:

$$\begin{aligned} &16\frac{1}{(q_\iota;q_\iota)_\infty^6}[\mathscr{R}_A]^7\mathscr{R}_D\frac{1}{\mathscr{R}_B} + 32\frac{1}{(q_\iota;q_\iota)_\infty^6}\frac{1}{\mathscr{R}_D}\frac{(q_\iota^2;q_\iota^2)_\infty(q_\iota^4;q_\iota^4)_\infty^{10}}{(q_\iota;q_\iota)_\infty^3} \\ &= 48\frac{1}{(q_\iota;q_\iota)_\infty(q_\iota^2;q_\iota^2)_\infty^{10}}\frac{1}{[\mathscr{R}_D]^5}[\mathscr{R}_B]^{17} + 448q_\iota[\mathscr{R}_B]^{13}\frac{1}{[\mathscr{R}_D]}\frac{1}{(q_\iota;q_\iota)_\infty(q_\iota^2;q_\iota^2)_\infty^{10}} \\ &+ 1280q_\iota^2[\mathscr{R}_B]^7\frac{(q_\iota^4;q_\iota^4)_\infty(q_\iota^8;q_\iota^8)_\infty^6}{(q_\iota;q_\iota)_\infty(q_\iota^2;q_\iota^2)_\infty^{12}} + 1024q_\iota^3\mathscr{R}_B\frac{(q_\iota^4;q_\iota^4)_\infty(q_\iota^8;q_\iota^8)_\infty^{14}}{(q_\iota;q_\iota)_\infty(q_\iota^2;q_\iota^2)_\infty^{14}}. \end{aligned} \tag{3.9}$$

Again, by using (3.5) and (3.7), we obtain:

$$\begin{aligned} &16\frac{[\mathscr{R}_A]^7\mathscr{R}_D}{(q_\iota;q_\iota)_\infty^6\mathscr{R}_B} + 32\frac{(q_\iota^2;q_\iota^2)_\infty^6[\mathscr{R}_B]^5}{(q_\iota;q_\iota)_\infty^6(q_\iota;q_\iota)_\infty^3\mathscr{R}_D} = 48\frac{[\mathscr{R}_B]^{17}}{(q_\iota;q_\iota)_\infty^6[\mathscr{R}_A]^5[\mathscr{R}_D]^5} + 448q_\iota\frac{[\mathscr{R}_B]^{13}}{(q_\iota;q_\iota)_\infty^6[\mathscr{R}_B]^5[\mathscr{R}_D]} \\ &+ 1280q_\iota^2\frac{(q_\iota^4;q_\iota^4)_\infty^4[\mathscr{R}_B]^7[\mathscr{R}_D]^3}{(q_\iota;q_\iota)_\infty(q_\iota^2;q_\iota^2)_\infty^{12}} + 1024q_\iota^3\mathscr{R}_B\frac{(q_\iota^4;q_\iota^4)_\infty^8[\mathscr{R}_D]^7}{(q_\iota;q_\iota)_\infty(q_\iota^2;q_\iota^2)_\infty^{14}}. \end{aligned} \tag{3.10}$$

After that, we get:

$$\begin{aligned} &16\frac{[\mathscr{R}_A]^7\mathscr{R}_D]}{(q_\iota;q_\iota)_\infty^6\mathscr{R}_B} + 32\frac{[\mathscr{R}_A]^3[\mathscr{R}_B]^5}{(q_\iota;q_\iota)_\infty^6\mathscr{R}_D} = 48\frac{[\mathscr{R}_B]^{17}}{(q_\iota;q_\iota)_\infty^6[\mathscr{R}_A]^5[\mathscr{R}_D]^5} + 448q_\iota\frac{[\mathscr{R}_B]^{13}}{(q_\iota;q_\iota)_\infty^6[\mathscr{R}_A]^5\mathscr{R}_D} \\ &+ 1280q_\iota^2\frac{[\mathscr{R}_B]^9[\mathscr{R}_D]^3}{(q_\iota;q_\iota)_\infty^6[\mathscr{R}_A]^5} + 1024q_\iota^3\frac{[\mathscr{R}_B]^5[\mathscr{R}_D]^7}{(q_\iota;q_\iota)_\infty^6[\mathscr{R}_A]^5}. \end{aligned} \tag{3.11}$$

□

Theorem 3.2. *If $\mathscr{R}_m$ $\mathscr{R}_\beta$ and $\mathscr{R}_\alpha$ are defined by $\mathscr{R}$-function equation (1.23). Then, the following relation holds:*

$$4\frac{[\mathscr{R}_A]^2[\mathscr{R}_B]^2(-q_\iota;q_\iota)_\infty^2}{(q_\iota;q_\iota)_\infty^2\mathscr{R}_D} = 4\frac{[\mathscr{R}_B]^8}{(q_\iota;q_\iota)_\infty^3\mathscr{R}_A[\mathscr{R}_D]^3} + 16q_\iota\frac{[\mathscr{R}_B]^4\mathscr{R}_D}{(q_\iota;q_\iota)_\infty^3\mathscr{R}_A}. \tag{3.12}$$

Proof. By using equation (2.7) and splitting the power of $U_{i's}$:

$$4\frac{U_2^4U_4^5}{U_1^6U_8^2} = 4\frac{U_4^{19}}{U_1^2U_2^{10}U_8^6} + 16q\frac{U_4^7U_8^2}{U_1^2U_2^6}.$$

Converting the above equation in q_ι-pochhammer symbols.

$$4\frac{(q_\iota^2;q_\iota^2)_\infty^4(q_\iota^4;q_\iota^4)_\infty^5}{(q_\iota;q_\iota)_\infty^6(q_\iota^8;q_\iota^8)_\infty^2} = 4\frac{(q_\iota^4;q_\iota^4)_\infty^{19}}{(q_\iota;q_\iota)_\infty^2(q_\iota^2;q_\iota^2)_\infty^{10}(q_\iota^8;q_\iota^8)_\infty^6} + 16q_\iota\frac{(q_\iota^4;q_\iota^4)_\infty^7(q_\iota^8;q_\iota^8)_\infty^2}{(q_\iota;q_\iota)_\infty^2(q_\iota^2;q_\iota^2)_\infty^6}. \tag{3.13}$$

$$4\frac{(q_\iota^2;q_\iota^2)_\infty^4(q_\iota^4;q_\iota^4)_\infty^5(-q_\iota;q_\iota)_\infty^4}{(q_\iota;q_\iota)_\infty^2(q_\iota^8;q_\iota^8)_\infty^2(q_\iota;q_\iota)_\infty^2(-q_\iota;q_\iota)_\infty^2} = 4\frac{(q_\iota^4;q_\iota^4)_\infty^{18}(q_\iota;q_\iota^2)_\infty(q_\iota^2;q_\iota^4)_\infty(q_\iota^4;q_\iota^8)_\infty}{(q_\iota;q_\iota)_\infty^3(q_\iota^2;q_\iota^2)_\infty^{10}(q_\iota^8;q_\iota^8)_\infty^4} + 16q_\iota\frac{(q_\iota^4;q_\iota^4)_\infty^8(q_\iota^8;q_\iota^8)_\infty^2}{(q_\iota;q_\iota)_\infty^3(q_\iota^2;q_\iota^2)_\infty^6}. \tag{3.14}$$

By using (3.4), (3.5) and (3.7), we get: Thus, by applying (3.4), (3.5) and (3.7), we get:

$$4\frac{[\mathscr{R}_A]^2[\mathscr{R}_B]^2(-q_\iota;q_\iota)_\infty^2}{(q_\iota;q_\iota)_\infty^2\mathscr{R}_D} = 4\frac{[\mathscr{R}_B]^8}{(q_\iota;q_\iota)_\infty^3\mathscr{R}_A[\mathscr{R}_D]^3} + 16q_\iota\frac{[\mathscr{R}_B]^4\mathscr{R}_D}{(q_\iota;q_\iota)_\infty^3\mathscr{R}_A}. \tag{3.15}$$

□

Theorem 3.3. *If $\mathscr{R}_m$ $\mathscr{R}_\beta$ and $\mathscr{R}_\alpha$ are defined by $\mathscr{R}$-function equation (1.23). Then, the following relation holds:*

$$\begin{aligned}4\frac{[\mathscr{R}_A]^2[\mathscr{R}_B]^2(-q_\iota;q_\iota)_\infty^2}{(q_\iota;q_\iota)_\infty^2\mathscr{R}_D} &= 4\frac{[\mathscr{R}_B]^9}{(q_\iota;q_\iota)_\infty^3[\mathscr{R}_A]^3\mathscr{R}_D\mathscr{R}_E} + 8q_\iota\frac{[\mathscr{R}_B]^9\mathscr{R}_E}{(q_\iota;q_\iota)_\infty^3[\mathscr{R}_A]^3[\mathscr{R}_D]^3}\\ &+ 16q_\iota\frac{[\mathscr{R}_B]^5[\mathscr{R}_D]^3}{(q_\iota;q_\iota)_\infty^3[\mathscr{R}_A]^3\mathscr{R}_E} + 32q_\iota^2\frac{[\mathscr{R}_B]^5\mathscr{R}_D}{(q_\iota;q_\iota)_\infty^3[\mathscr{R}_A]^3\mathscr{R}_E}.\end{aligned} \tag{3.16}$$

Proof. Converting the equation (2.8) in q_ι-pochhammer symbols:

$$\begin{aligned}4\frac{(q_\iota^2;q_\iota^2)_\infty^4(q_\iota^4;q_\iota^4)_\infty^5}{(q_\iota;q_\iota)_\infty^6(q_\iota^8;q_\iota^8)_\infty^2} &= 4\frac{(q_\iota^4;q_\iota^4)_\infty^{19}}{(q_\iota;q_\iota)_\infty^2(q_\iota^2;q_\iota^2)_\infty^{10}(q_\iota^{16};q_\iota^{16})_\infty^2} + 8q_\iota\frac{(q_\iota^4;q_\iota^4)_\infty^{21}(q_\iota^{16};q_\iota^{16})_\infty^2}{(q_\iota^2;q_\iota^2)_\infty^{15}(q_\iota^8;q_\iota^8)_\infty^7} +\\ &16q_\iota\frac{(q_\iota^4;q_\iota^4)_\infty^7(q_\iota^8;q_\iota^8)_\infty^7}{(q_\iota^2;q_\iota^2)_\infty^{11}(q_\iota^{16};q_\iota^{16})_\infty^2} + 32q_\iota^2\frac{(q_\iota^4;q_\iota^4)_\infty^9(q_\iota^8;q_\iota^8)_\infty(q_\iota^{16};q_\iota^{16})_\infty^2}{(q_\iota^2;q_\iota^2)_\infty^{11}}.\end{aligned} \tag{3.17}$$

Equation (3.17) can be written as:

$$\begin{aligned}4\frac{(q_\iota^2;q_\iota^2)_\infty^4}{(q_\iota;q_\iota)_\infty^2}\frac{(q_\iota^4;q_\iota^4)_\infty}{(q_\iota^8;q_\iota^8)_\infty^2}\frac{(q_\iota^4;q_\iota^4)_\infty^4}{(q_\iota;q_\iota)_\infty^4} &= 4\frac{(q_\iota^4;q_\iota^4)_\infty^{18}}{(q_\iota^2;q_\iota^2)_\infty^9}\frac{(q_\iota^4;q_\iota^4)_\infty}{(q_\iota;q_\iota)_\infty^2(q_\iota^2;q_\iota^2)_\infty(q_\iota^{16};q_\iota^{16})_\infty^2} + 8q_\iota\frac{(q_\iota^4;q_\iota^4)_\infty^{20}}{(q_\iota^2;q_\iota^2)_\infty^{10}}\\ \frac{(q_\iota^{16};q_\iota^{16})_\infty^2}{(q_\iota^8;q_\iota^8)_\infty}\frac{(q_\iota^4;q_\iota^4)_\infty}{(q_\iota^8;q_\iota^8)_\infty^2}&\frac{1}{(q_\iota^2;q_\iota^2)_\infty^5(q_\iota^8;q_\iota^8)_\infty^4} + 16q_\iota\frac{(q_\iota^4;q_\iota^4)_\infty^6}{(q_\iota^2;q_\iota^2)_\infty^3}\frac{(q_\iota^8;q_\iota^8)_\infty}{(q_\iota^{16};q_\iota^{16})_\infty^2}\frac{(q_\iota^4;q_\iota^4)_\infty(q_\iota^8;q_\iota^8)_\infty^6}{(q_\iota^2;q_\iota^2)_\infty^7}\\ + 32q_\iota^2\frac{(q_\iota^4;q_\iota^4)_\infty^8}{(q_\iota^2;q_\iota^2)_\infty^4}&\frac{(q_\iota^4;q_\iota^4)_\infty(q_\iota^8;q_\iota^8)_\infty(q_\iota^{16};q_\iota^{16})_\infty^2}{(q_\iota^2;q_\iota^2)_\infty^7}.\end{aligned} \tag{3.18}$$

By using (3.4), (3.5), (3.7) and (3.8), we get the desired results. □

Theorem 3.4.

$$8\frac{[\mathscr{R}_A]^3\mathscr{R}_D}{(q_\iota;q_\iota)_\infty^3} = 8\frac{[\mathscr{R}_B]^6}{(q_\iota;q_\iota)_\infty^3\mathscr{R}_A\mathscr{R}_D} + 32q_\iota\frac{[\mathscr{R}_B]^2[\mathscr{R}_D]^3}{(q_\iota;q_\iota)_\infty^3\mathscr{R}_A}. \tag{3.19}$$

Proof. Converting the equation (2.9) in q_ι-pochhammer symbols:

$$8\frac{(q_\iota^2;q_\iota^2)_\infty^6(q_\iota^8;q_\iota^8)_\infty^2}{(q_\iota;q_\iota)_\infty^6(q_\iota^4;q_\iota^4)_\infty} = 8\frac{(q_\iota^4;q_\iota^4)_\infty^{13}}{(q_\iota;q_\iota)_\infty^2(q_\iota^2;q_\iota^2)_\infty^8(q_\iota^8;q_\iota^8)_\infty^2} + 32q_\iota\frac{(q_\iota^4;q_\iota^4)_\infty(q_\iota^8;q_\iota^8)_\infty^6}{(q_\iota;q_\iota)_\infty^2(q_\iota^2;q_\iota^2)_\infty^4}. \tag{3.20}$$

By using (3.4), (3.5), (3.7) and (3.8), we get the desired results. □

Theorem 3.5.

$$\begin{aligned}&108\frac{[\mathscr{R}_A]^7(q_\iota^3;q_\iota^3)_\infty^6}{(q_\iota;q_\iota)_\infty^{11}\mathscr{R}_C}+324q_\iota\frac{[\mathscr{R}_A]^3[\mathscr{R}_C]^3(q_\iota^3;q_\iota^3)_\infty^6}{(q_\iota;q_\iota)_\infty^{11}}=108\frac{[\mathscr{R}_A]^3[\mathscr{R}_B]^7(q_\iota^3;q_\iota^3)_\infty^6}{(q_\iota;q_\iota)_\infty^{11}\mathscr{R}_B\mathscr{R}_C[\mathscr{R}_D]^2}+\\&324q_\iota\frac{[\mathscr{R}_B]^7[\mathscr{R}_C]^3(q_\iota^3;q_\iota^3)_\infty^6}{(q_\iota;q_\iota)_\infty^{11}\mathscr{R}_A\mathscr{R}_B[\mathscr{R}_D]^2}+432q_\iota\frac{[\mathscr{R}_A]^3[\mathscr{R}_B]^2[\mathscr{R}_D]^2(q_\iota^3;q_\iota^3)_\infty^6}{(q_\iota;q_\iota)_\infty^{11}\mathscr{R}_C}+\\&1296q_\iota^2\frac{[\mathscr{R}_B]^2[\mathscr{R}_C]^3[\mathscr{R}_D]^2(q_\iota^3;q_\iota^3)_\infty^6}{(q_\iota;q_\iota)_\infty^{11}\mathscr{R}_A}.\end{aligned}\tag{3.21}$$

Proof. Convert (2.10) in q_ι-pochhammer symbols and employ the similar approach as in previous theorems and then use (3.4), (3.5), (3.6) and (3.7) to get the desired result. □

4. Connections with Combinatorial Partition-Theoretic Identities

Recent advances in partition theory have revealed deep connections between q_ι-series identities and combinatorial partition functions, as demonstrated in [13, 24, 25]. These developments often employ generating function techniques and combinatorial interpretations [10], building bridges between previously distinct areas of partition theory.

Significant progress was made by Andrews [1], who related Schur's, Göllnitz-Gordon, and Göllnitz partitions through multivariate $\mathscr{R}$-functions. This work was later extended by Srivastava [19], who introduced refined versions $\mathscr{R}_\alpha, \mathscr{R}_\beta$ and $\mathscr{R}_m$ of these functions. Building on these foundations, we develop a new generalization of $\mathscr{R}$-functions using eighth-order mock theta functions. Our approach yields:

- Novel identities connecting mock theta functions to partition theory
- Enhanced understanding of classical q_ι-series through combinatorial lenses
- New tools for analytic and combinatorial investigations of partitions

5. Conclusion

This study contributes to advances in recent studies by introducing a generalization of the $\mathscr{R}$-function derived from eighth-order mock theta function identities. Using the extensive framework of these functions, we derive significant results that further advance our understanding of classical q_ι-series identities while uncovering previously unexplored connections in partition theory.

Our findings give a novel perspective for examining the correlation between mock theta functions and extended $\mathscr{R}$-functions. These contributions yield valuable tools and identities applicable to both analytic and combinatorial approaches in partition theory, paving the way for further investigations.

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